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Multiscale reduction of discrete nonlinear Schrödinger equations

D Levi[1], **M Petrera**[2] **and C Scimiterna**[2]

[1] Dipartimento di Ingegneria Elettronica, Università degli Studi Roma Tre and Sezione INFN, Roma Tre Via della Vasca Navale 84, 00146 Roma, Italy
[2] Dipartimento di Fisica Università degli Studi Roma Tre Via della Vasca Navale 84, 00146 Roma, Italy

E-mail: levi@fis.uniroma3.it, petrera@fis.uniroma3.it and scimiterna@fis.uniroma3.it

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Abstract
We use a discrete multiscale analysis to study the asymptotic integrability of differential–difference equations. In particular, we show that multiscale perturbation techniques provide an analytic tool to derive necessary integrability conditions for two well-known discretizations of the nonlinear Schrödinger equation.

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1. Introduction

The nonlinear Schrödinger (NLS) equation,

$$\mathrm{i}\partial_t f + \partial_{xx} f = \sigma |f|^2 f, \qquad f = f(x,t), \qquad \sigma = \pm 1, \tag{1}$$

is a *universal* nonlinear integrable partial differential equation for models with weak nonlinear effects. Here, x is the spatial variable and t is the time, while ∂ denotes differentiation with respect to its subscript. It has been central for almost 40 years in many different scientific areas, and it appears in several physical contexts; see for instance [4, 5, 29].

In [33], Zakharov and Shabat proved its integrability by solving its associated spectral problem. From the integrability of equation (1) it follows the existence of infinitely many symmetries and conservation laws and the solvability of its associated Cauchy problem. In correspondence with its symmetries one finds an infinite number of exact solutions, the solitons, which, up to a phase, emerge unperturbed from the interaction among themselves.

The problem of the discretization of the NLS equation has been the subject of an intensive research. In literature, one may find a few discretizations of the NLS equation. An integrable differential–difference equation discretizing equation (1) has been found by Ablowitz and Ladik [2]. It reads

$$\mathrm{i}\partial_t f_n + \frac{f_{n+1} - 2f_n + f_{n-1}}{2h^2} = \sigma |f_n|^2 \frac{f_{n+1} + f_{n-1}}{2}, \tag{2}$$

where $n \in \mathbb{Z}$ and $0 < h < 1$ is a parameter related to the space discretization. As one can easily see, in the limit $h \to 0$, equation (2) goes into the NLS equation (1). Equation (2) admits a Lax pair and consequently it has an infinite number of generalized symmetries and local conservation laws, which provide explicit soliton solutions [4].

From the applications' point of view of, the most relevant differential–difference NLS equation is given by

$$\mathrm{i}\partial_t f_n + \frac{f_{n+1} - 2f_n + f_{n-1}}{2h^2} = \sigma |f_n|^2 f_n. \tag{3}$$

Equation (3) is one of the most studied lattice models (see for instance [1, 4, 9, 13–15] and references therein). Its study has a long and fascinating history, beginning in the 1950s in solid state physics with Holstein's model for polaron motion in molecular crystals [19] and later appears in biophysics with Davydov's model for energy transport in biomolecules [28]. Among the many recent applications of equation (3), let us just mention the theory of Bose–Einstein condensates in optical lattices [1] and semiconductors [15]. Its continuous limit goes again into the integrable NLS equation (1). The discrete NLS equation (3) possesses exact discrete breather solutions [14], and just a few number of conserved quantities and symmetries are known. Numerical schemes have been used to exhibit its chaotic behavior [3]. A proof of its non-integrability, based on multiscale techniques, has been recently presented by the authors in [26].

By introducing the parameter $s = 0, 1$, the discrete NLS equations (2)–(3) may be combined in the equation

$$\mathrm{i}\partial_t f_n + \frac{(f_{n+1} - 2f_n + f_{n-1})\left(1 - s\sigma h^2 |f_n|^2\right)}{2h^2} = \sigma |f_n|^2 f_n. \tag{4}$$

The case $s = 1$ corresponds to equation (2), while the case $s = 0$ gives equation (3).

Multiscale analysis [30, 31] is an important perturbative method for finding approximate solutions to many physical problems by reducing a given partial differential equation to a simpler equation, which can be integrable [8]. Multiscale expansions are structurally strong and can be applied to both integrable and non-integrable systems. Zakharov and Kuznetsov [32] have shown that, starting from an integrable partial differential equation and performing a proper multiscale expansion, one may obtain other integrable systems. In particular, they showed that the slow-varying amplitude of a dispersive wave solution of equation (1) satisfies the Korteweg–de Vries (KdV) equation, and vice versa.

Let us give a sketch of their derivation, showing how to obtain the KdV equation as the lowest order of the multiscale expansion of the NLS equation (1) with $\sigma = 1$, the so-called repulsive NLS equation. To do so one separates the complex field f in its amplitude and phase, $f(x,t) = [\nu(x,t)/2]^{1/2} \exp[\mathrm{i}\phi(x,t)]$ and rewrite the NLS equation as the system of the real partial differential equations

$$\partial_t \nu_t + \partial_x(\nu\varphi) = 0, \tag{5}$$

$$\partial_t \varphi + \varphi\partial_x\varphi + \partial_x \nu = \tfrac{1}{2}\partial_x \nu^{-1/2}\partial_x^2 \nu^{1/2}, \tag{6}$$

where $\varphi = \partial_x \phi$. For long waves and small perturbations around the equilibrium solution of equation (1), we can define the following formal perturbation expansions:

$$v(x, t) = 1 + \sum_{i=1}^{\infty} \epsilon^{2i} v^{(i)}(x', t'), \tag{7}$$

$$\phi(x, t) = -t + \sum_{i=1}^{\infty} \epsilon^{2i-1} \phi^{(i)}(x', t'), \tag{8}$$

where ϵ is a small perturbation parameter and $x' = \epsilon(x - t)$ and $t' = \epsilon^3 t$ are suitable *slow* variables. By inserting expansions (7) and (8) into equations (5) and (6), a direct computation shows that the lowest nontrivial order of the perturbation, that is ϵ^3, provides an evolution equation for the field $v^{(1)}$ wrt the slow time t':

$$\partial_{t'} v^{(1)} + \tfrac{3}{2} v^{(1)} \partial_{x'} v^{(1)} - \tfrac{1}{8} \partial_{x'}^3 v^{(1)} = 0,$$

that is a KdV equation.

In [8], Calogero and Eckhaus have used the multiscale technique, at its lowest nontrivial order, as a tool to give necessary conditions for the integrability of large classes of partial differential equations both in 1+1 and 2+1 dimensions. In particular, it has been shown that the non-integrability of the resulting multiscale reduction is a consequence of the non-integrability of the ancestor system. The derivation of the higher order terms of multiscale expansions has been carried out by Degasperis, Manakov and Santini in [11] and Kodama and Mikhailov in [22]. In [12], Degasperis and Procesi introduced the notion of *asymptotic integrability* of order n by requiring that the multiscale expansion be verified up to order n of the perturbation parameter. An integrable partial differential equation, as the NLS equation (1), has an asymptotic integrability of infinite order.

Some attempts to extend this approach to discrete equations have been proposed [6, 16–18, 23–25]. In [23–25], a multiscale technique for dispersive $\mathbb{Z}^2$ lattice equations has been developed which is based on the dilation transformations of discrete shift operators. To our knowledge, the dilation of the lattice has been carried out for the first time by Jordan [20]. Let us illustrate the basic procedure in the case of a function $f_n : \mathbb{Z} \to \mathbb{C}$ depending only on one discrete index. Let T_n be the shift operator, $T_n f_n = f_{n+1}$ and $\Delta_n = T_n - 1$ be the difference operator of order 1. The difference of order j in a new discrete variable n' is expressed in terms of an infinite number of differences on the lattice of the variable n by means of the following formula [20]:

$$\Delta_{n'}^{j} f_{n'} = \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} f_{n'+i} = \sum_{i=j}^{\infty} \frac{j!}{i!} \sum_{k=j}^{i} \omega^k \mathfrak{S}_i^k \mathcal{S}_k^j \Delta_n^i f_n, \tag{9}$$

where ω is the ratio of the increment in the lattice of the variable n, wrt the variable n' and the coefficients $\mathfrak{S}_i^k$ and $\mathcal{S}_k^j$ which are the Stirling numbers of the first and second kinds, respectively.

The Jordan formula (9) implies that a rescaling of a lattice variable gives rise to nonlocal results. Therefore, to avoid the presence of infinite sums one needs to truncate the series (9), namely to introduce a *slow-varying condition*:

$$\Delta_n^{p+1} f_n = 0, \tag{10}$$

where p being a positive integer. A more general slow-varying condition has recently been introduced in [27].

In [23–25], the multiscale analysis has been performed taking into account condition (10). As a consequence, the reduced discrete equations turned out to be non-integrable even if the ancestor equation was integrable. However, as shown in [16–18], if $p = \infty$ the reduced equations become formally continuous, and their integrability may be properly preserved by the multiscale procedure. In this way, multiscale techniques easily fit with both difference–difference and differential–difference equations. The results contained in [16–18] confirm a discrete analog of the Zakharov–Kuznetsov claim [32]: 'if a nonlinear dispersive discrete equation is integrable then its lowest order multiscale reduction is an integrable NLS equation'.

In this paper, we present the multiscale perturbation analysis of equation (4), thus extending to the discrete setting the approach used in [11, 12, 22, 32]. The derivation of the higher order terms in the perturbation expansion will enable us to provide an analytic evidence of the non-integrability of equation (3). In fact, even if its lowest order reduction is an integrable KdV-type equation, the higher order reductions exhibit non-integrable behaviors (see also our recent letter [26] where no details were presented). In contrast, the same calculations for the case of equation (2) will show that the Ablowitz–Ladik discrete NLS equation satisfies all the integrability conditions up to the same order considered in the non-integrable case. This is an indication of its asymptotic integrability of finite order but not a proof of its integrability as for it we should go up to infinite order.

The paper is organized as follows. Section 2 is devoted to the presentation of some technical details and basic formulas for the multiscale analysis of equation (4). The main results of the perturbation analysis will be given in section 3. In the concluding section 4, we discuss further perspectives of this approach.

2. Basic formulas for multiscale analysis of discrete NLS equations

As for the continuous NLS equation, also for the discrete NLS equation (4), we introduce amplitude and phase of the function $f_n(t)$, namely $f_n(t) = [\nu_n(t)]^{1/2}\exp[\mathrm{i}\phi_n(t)]$. Therefore, the discrete NLS equation (4) may be written as the following nonlinear system of the real differential–difference equations ($s = 1$ for (2) and $s = 0$ for (3)):

$$\partial_t \nu_n = \left(s\sigma\nu_n - \frac{1}{h^2}\right)\left[\sqrt{\nu_n\nu_{n+1}}\sin(\phi_{n+1}-\phi_n) + \sqrt{\nu_n\nu_{n-1}}\sin(\phi_{n-1}-\phi_n)\right], \tag{11}$$

$$\partial_t\phi_n = -\frac{1}{h^2} + \frac{1}{2}\left[\frac{1}{h^2} + (s-2)\sigma\nu_n\right]\left[\sqrt{\frac{\nu_{n+1}}{\nu_n}}\cos(\phi_{n+1}-\phi_n) + \sqrt{\frac{\nu_{n-1}}{\nu_n}}\cos(\phi_{n-1}-\phi_n)\right]. \tag{12}$$

By analogy with the continuous case, see equations (7)–(8); the real fields $\nu_n(t)$ and $\phi_n(t)$ are expanded around the constant solution $f_n(t) = \exp(-\mathrm{i}\sigma t)$ in the following way:

$$\nu_n(t) = 1 + \sum_{i=1}^{\infty} \epsilon^{2i}\nu^{(i)}(\kappa, \{t_m\}_{m\geqslant 1}), \tag{13}$$

$$\phi_n(t) = -\sigma t + \sum_{i=1}^{\infty} \epsilon^{2i-1}\phi^{(i)}(\kappa, \{t_m\}_{m\geqslant 1}), \tag{14}$$

where ϵ, with $0 < \epsilon \ll 1$, is the perturbation parameter. The fields $\nu^{(i)}$ and $\phi^{(i)}$ in equations (13)–(14) depend on the slow-space variable $\kappa = \epsilon\zeta n, \zeta \in \mathbb{R}$, and the slow-time variables $t_m = \epsilon^{2m-1}t, m \geqslant 1$. The free parameter ζ will be fixed later so as to obtain a suitable continuous limit.

In general, given a function $u_n(t) = v(\kappa, \{t_m\}_{m\geqslant 1})$, we expand $u_{n\pm 1}(t)$ and $\partial_t u_n(t)$ in terms of the slow variables κ and $\{t_m\}_{m\geqslant 1}$ (see [16, 17] for further details). Let T_n be the shift operator defined by $T_n^{\pm} u_n = u_{n\pm 1}$. Then we have

$$u_{n\pm 1} = \left(T_\kappa^{\pm}\right)^{\epsilon\zeta} v(\kappa, \{t_m\}_{m\geqslant 1}) = \sum_{i=0}^{\infty} \frac{(\pm\epsilon\zeta\delta_\kappa)^i}{i!} v(\kappa, \{t_m\}_{m\geqslant 1}), \tag{15}$$

with

$$\delta_\kappa = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} \Delta_\kappa^i, \qquad \Delta_\kappa^i = (T_\kappa - 1)^i \tag{16}$$

and

$$\partial_t u_n = \sum_{i=1}^{\infty} \epsilon^{2i-1} \partial_{t_i} v(\kappa, \{t_m\}_{m\geqslant 1}). \tag{17}$$

If u_n is a slow-varying function of order p, see equation (10), we can truncate the infinite series in equation (16). In such a case, the δ_κ operators reduce to polynomials in the Δ_κ operators of order at most p. Hereafter, we shall assume that $p = \infty$ and the δ_κ operators are formal differential operators.

Taking into account expansions (13)–(14) and equations (15) and (17), we have the following formulas for the shifts of the functions $\nu_n(t)$ and $\phi_n(t)$:

$$\nu_{n\pm 1} = 1 + \sum_{j=2}^{\infty} \epsilon^j \sum_{i=1}^{[j/2]} \frac{(\pm\zeta\delta_\kappa)^{j-2i}}{(j-2i)!} \nu^{(i)}(\kappa, \{t_m\}_{m\geqslant 1}), \tag{18}$$

$$\phi_{n\pm 1} = -\sigma t + \sum_{j=1}^{\infty} \epsilon^j \sum_{i=1}^{[(j+1)/2]} \frac{(\pm\zeta\delta_\kappa)^{j-2i+1}}{(j-2i+1)!} \phi^{(i)}(\kappa, \{t_m\}_{m\geqslant 1}), \tag{19}$$

and their time derivatives,

$$\partial_t \nu_n = \sum_{j=2}^{\infty} \epsilon^{2j-1} \sum_{i=1}^{j-1} \partial_{t_i} \nu^{(j-i)}(\kappa, \{t_m\}_{m\geqslant 1}), \tag{20}$$

$$\partial_t \phi_n = -\sigma + \sum_{j=1}^{\infty} \epsilon^{2j} \sum_{i=1}^{j} \partial_{t_i} \phi^{(j-i+1)}(\kappa, \{t_m\}_{m\geqslant 1}). \tag{21}$$

3. Main results

The multiscale analysis of the system of the real differential–difference equations (11)–(12) is carried out by inserting the formal expansions (13)–(14) and (18)–(21) into equations (11)–(12) and by requiring that the resulting equations be satisfied at all orders in ϵ.

The lowest non-trivial order corresponds to ϵ^2. It gives

$$\nu^{(1)} = -\sigma \partial_{t_1} \phi^{(1)}.$$

From this point onwards, all results will only be presented for the phase functions $\phi^{(i)}, i \geqslant 1$, since any $\nu^{(i)}, i > 1$, is obtained from the even perturbation orders and may be expressed in terms of the $\phi^{(j)}$ s with $j \leqslant i$ and their derivatives.

At order ϵ^3 we get

$$\left(\partial_{t_1}^2 - c^2\delta_\kappa^2\right)\phi^{(1)} = 0, \qquad c = \pm\frac{\zeta(\sigma - sh^2)^{1/2}}{h}.$$

As c has to be real, our multiscale analysis is performed only for $\sigma = 1$. Moreover, we choose $\zeta = h$, so that $c = \pm(1 - sh^2)^{1/2}$ remains finite as $h \to 0$. Therefore, the asymptotically bounded solution of the resulting equation at this order is given by $\phi^{(1)} = \phi^{(1)}(x, \{t_m\}_{m\geqslant 2})$ with $x = \kappa - ct_1$.

At order ϵ^5, the no-secular term condition implies $\left(\partial_{t_1}^2 - c^2\delta_\kappa^2\right)\phi^{(2)} = 0$, so that $\phi^{(2)} = \phi^{(2)}(x, \{t_m\}_{m\geqslant 2})$. At this same order, the evolution equation for $\phi^{(1)}$ wrt the slow time t_2 reads

$$\partial_{t_2}\phi^{(1)} = K_2[\phi^{(1)}], \qquad K_2[\phi^{(1)}] = \alpha_1\partial_x^3\phi^{(1)} + \alpha_2(\partial_x\phi^{(1)})^2, \tag{22}$$

with

$$\alpha_1 = \frac{c}{24}[3 - (3s+1)h^2], \qquad \alpha_2 = sh^2 - \frac{3}{4}.$$

Equation (22) is a potential KdV equation and $K_2\left[\phi^{(1)}\right]$ is the second flow of the integrable hierarchy associated with the potential KdV equation. A necessary condition for the integrability of the system (11)–(12) is that its multiscale reductions provide the integrable evolution equations ($j \geqslant 3$):

$$\partial_{t_j}\phi^{(1)} = K_j[\phi^{(1)}] = \beta_j \int^x \mathrm{d}u\, \mathcal{L}^{j-1}\left[\partial_u^2\phi^{(1)}\right], \tag{23}$$

where $\mathcal{L}$ is the recursive operator associated with the KdV hierarchy,

$$\mathcal{L}[f(x)] = \partial_x^2 f(x) - \frac{\partial_x\phi^{(1)}}{\alpha_1}f(x) - \frac{\partial_x^2\phi^{(1)}}{2\alpha_1}\int^x \mathrm{d}u\, f(u),$$

and the β_j s are the real coefficients to be fixed.

According to a general procedure for the multiscale analysis of the partial differential equations [10–12], we now assign a formal degree to the x derivatives of the functions $\phi^{(j)}$:

$$\deg\left(\partial_x^\ell\phi^{(j)}\right) = \ell + 2j - 1, \qquad \ell \geqslant 0,$$

and define $\mathcal{P}_n$ as the vector space spanned by the products of all derivatives $\partial_x^\ell\phi^{(j)}$ with total degree n. We denote by $\mathcal{P}_n^{(r)} \subset \mathcal{P}_n$ the subspace spanned by those products of derivatives $\partial_x^\ell\phi^{(j)}$ with $j \leqslant r$.

After caring for secularities, the order ϵ^7 yields $\phi^{(3)} = \phi^{(3)}(x, \{t_m\}_{m\geqslant 2})$ and the following non-homogeneous evolution equation for the field $\phi^{(2)}$ wrt the slow time t_2, depending on $\phi^{(1)}$ and its derivatives:

$$\begin{aligned}\partial_{t_2}\phi^{(2)} - \alpha_1\partial_x^3\phi^{(2)} - 2\alpha_2\partial_x\phi^{(1)}\partial_x\phi^{(2)} &= -\partial_{t_3}\phi^{(1)} + \alpha_3\left(\partial_x^2\phi^{(1)}\right)^2 + \alpha_4(\partial_x\phi^{(1)})^3\\ &\quad + \alpha_5\partial_x\phi^{(1)}\partial_x^3\phi^{(1)} + \alpha_6\partial_x^5\phi^{(1)},\end{aligned} \tag{24}$$

where

$$\alpha_3 = \frac{h^2[16h^2s - 5(1+3s)] + 7}{64}, \qquad \alpha_4 = \frac{ch^2(1+7s)}{12},$$

$$\alpha_5 = \frac{h^2[16h^2s - 3(3+s)] - 3}{48}, \qquad \alpha_6 = -\frac{c[h^4(15s+1) + 30h^2(s-1) - 15]}{1920}.$$

Substituting equation (23) with $j = 3$ into equation (24) and fixing $\beta_3 = -\alpha_6$ in order to remove residual secularities, equation (24) reduces to the following evolution equation for the field $\phi^{(2)}$ wrt the slow time t_2:

$$\partial_{t_2}\phi^{(2)} - K_2'[\phi^{(1)}]\phi^{(2)} = f^{(t_2)}, \tag{25}$$

where $K'_j[\phi^{(1)}]\psi$ is the Fréchet derivative of the flow $K_j[\phi^{(1)}]$ along the direction ψ,

$$K'_j[\phi^{(1)}]\psi = \frac{\mathrm{d}}{\mathrm{d}r} K_j[\phi^{(1)} + r\psi]\bigg|_{r=0}.$$

In equation (25), the forcing term $f^{(t_2)}$ is a well-defined element of $\mathcal{P}_6^{(1)}$, $\dim \mathcal{P}_6^{(1)} = 3$, namely a linear combination of three independent differential monomials (see the appendix, equation (A.3)), with known coefficients which are the polynomial functions of h.

Now the request for the integrability of (11)–(12) implies the existence of the following evolution equation for the field $\phi^{(2)}$ wrt the slow time t_3:

$$\partial_{t_3}\phi^{(2)} - K'_3(\phi^{(1)})\phi^{(2)} = f^{(t_3)}, \tag{26}$$

where $f^{(t_3)} \in \mathcal{P}_8^{(1)}$, $\dim \mathcal{P}_8^{(1)} = 6$ (see the appendix, equation (A.4)). Hence, the following compatibility condition must hold:

$$\{\partial_{t_3} - K'_3[\phi^{(1)}]\} f^{(t_2)} = \{\partial_{t_2} - K'_2[\phi^{(1)}]\} f^{(t_3)}. \tag{27}$$

Such a condition allows us to express the coefficients of the polynomial $f^{(t_3)}$ in terms of those of $f^{(t_2)}$, and it does not impose any further constraint on the coefficients of $f^{(t_2)}$ (see the appendix, equation (A.5)). As in our case, this condition is satisfied, we conclude that the nonlinear system (11)–(12) has an asymptotic integrability of order 7 irrespective of the value of s.

The next perturbation order, that is ϵ^9, gives rise to a bifurcation between the non-integrable ($s = 0$) and the integrable cases ($s = 1$). For the sake of clarity, we will study separately the two cases.

- *The case* $s = 0$. At the order ϵ^9, the resulting equations provide the evolution of the field $\phi^{(3)}$ wrt the slow time t_2. This is given by an integro-differential equation. To reduce it to a purely differential equation, we introduce the fields $\varphi^{(j)} = \partial_x\phi^{(j)}$. Taking care of secularities and taking into account that $\phi^{(1)}$ evolves wrt the slow time t_4 according to equation (23) with $j = 4$, we get $\phi^{(4)} = \phi^{(4)}(x, \{t_m\}_{m\geqslant 2})$ and

$$\partial_{t_2}\varphi^{(3)} - H'_2[\varphi^{(1)}]\varphi^{(3)} = g^{(t_2)}, \tag{28}$$

where $H'_j[\varphi^{(1)}]\psi$ is the Fréchet derivative along ψ of the jth KdV flow $H_j[\varphi^{(1)}] = \partial_x K_j[\varphi^{(1)}]$. Here $g^{(t_2)}$ is a known element of the space $\mathcal{P}_9^{(2)}$, $\dim \mathcal{P}_9^{(2)} = 14$ (see the appendix, equation (A.7)). The evolution equation of $\varphi^{(3)}$ wrt the slow time t_3 takes the form

$$\partial_{t_3}\varphi^{(3)} - H'_3[\varphi^{(1)}]\varphi^{(3)} = g^{(t_3)}, \tag{29}$$

where the coefficients of $g^{(t_3)} \in \mathcal{P}_{11}^{(2)}$, $\dim \mathcal{P}_{11}^{(2)} = 31$, are determined by requiring the compatibility condition

$$\{\partial_{t_3} - H'_3[\varphi^{(1)}]\} g^{(t_2)} = \{\partial_{t_2} - H'_2[\varphi^{(1)}]\} g^{(t_3)}. \tag{30}$$

Equation (30) is a necessary condition for the integrability of the system (11)–(12) with $s = 0$. In this case, only 9 out of the 14 coefficients of $g^{(t_2)}$ are independent. Thus, we have five integrability conditions (see the appendix for further details). It turns out that the obtained constraints on the polynomial $g^{(t_2)}$ are not satisfied by the coefficients computed in equation (28). Therefore, the system (11)–(12) with $s = 0$, namely the discrete NLS equation (3), does not fulfil the necessary conditions assuring its integrability.

- *The case* $s = 1$. In this case, the resulting equations are purely differential and one can remain within the potential KdV hierarchy. Taking care of secularities and taking into account that $\phi^{(1)}$ evolves wrt the slow time t_4 according to equation (23) with $j = 4$, we get $\phi^{(4)} = \phi^{(4)}(x, \{t_m\}_{m\geqslant 2})$ and

$$\partial_{t_2}\phi^{(3)} - K_2'[\phi^{(1)}]\phi^{(3)} = h^{(t_2)}, \tag{31}$$

where $h^{(t_2)}$ is a known element of the space $\mathcal{P}_8^{(2)}$, $\dim \mathcal{P}_8^{(2)} = 11$. The evolution equation of $\phi^{(3)}$ wrt the slow time t_3 takes the form

$$\partial_{t_3}\phi^{(3)} - K_3'[\varphi^{(1)}]\phi^{(3)} = h^{(t_3)}, \tag{32}$$

where the coefficients of $h^{(t_3)} \in \mathcal{P}_{10}^{(2)}$, $\dim \mathcal{P}_{10}^{(2)} = 24$, are determined by requiring the compatibility condition

$$\{\partial_{t_3} - K_3'[\phi^{(1)}]\}h^{(t_2)} = \{\partial_{t_2} - K_2'[\phi^{(1)}]\}h^{(t_3)}. \tag{33}$$

In such a case, it turns out that all the constraints imposed by (33) on the 11 coefficients of the polynomial $h^{(t_2)}$ are satisfied by the coefficients computed in equation (31) (see the appendix for further details). This proves that the system (11)–(12) with $s = 1$, namely the discrete NLS equation (2), has an asymptotic integrability of order 9. Actually, since the discrete NLS equation (2) is known to be integrable, its asymptotic integrability should be of order infinite.

The above results may be summarized in the following proposition.

Proposition 1. *The nonlinear differential–difference equation*

$$\mathrm{i}\partial_t f_n + \frac{f_{n+1} - 2f_n + f_{n-1}}{2h^2} = |f_n|^2 f_n,$$

is non-integrable. In particular, its multiscale reduction, carried out by using the formal expansions (13)–(14) and (18)–(21), shows that it has an asymptotic integrability of order 7.

The differential–difference Ablowitz–Ladik equation,

$$\mathrm{i}\partial_t f_n + \frac{f_{n+1} - 2f_n + f_{n-1}}{2h^2} = |f_n|^2 \frac{f_{n+1} + f_{n-1}}{2},$$

has an asymptotic integrability of order 9. (Actually its asymptotic integrability should be of order infinite since it is known to be integrable.)

4. Concluding remarks

The present paper has been devoted to the derivation of higher order terms of the multiscale perturbation of discrete NLS equations around the constant equilibrium solution. This enabled us to study the asymptotic integrability of equations (2)–(3), thus proving that the discrete NLS equation (3) is non-integrable. Such a result has been already established in [26], but a detailed presentation of the integrability conditions appears for the first time in the present paper. Moreover, we have also investigated the asymptotic integrability of the Ablowitz–Ladik discrete NLS equation (2).

We note that the obtained results can also be used to construct the approximate soliton solutions of the discrete NLS equations (2)–(3). They will be expressed in terms of the solutions of the continuous equations belonging to the KdV and potential KdV hierarchies. More precisely, the solutions of the lowest order term of the multiscale expansion of (2)–(3) will be expressed in terms of a soliton solution of the potential KdV equation.

It is worth noting that the presented discrete multiscale technique fits with both differential–difference and difference–difference equations. Therefore, it can be used to investigate the asymptotic integrability of a large class of discrete dynamical systems. The method turns out to be a useful analytic tool whenever one has to deal with a discrete equation whose integrability is not established yet.

As a future work, we plan to investigate the asymptotic integrability of the following differential–difference equations [7]:

$$\partial_t f_n + \frac{f_{n+1} - 2f_n + f_{n-1}}{2h^2} = f_n + g(f_{n-1}, f_n, f_{n+1}),$$

withg being a homogeneous polynomial of degree 3 and [21]

$$\partial_t f_n + \frac{f_{n+1} - 2f_n + f_{n-1}}{2h^2} = \frac{\sigma f_n |f_n|^2}{1 + \alpha |f_n|^2},$$

a saturable discrete NLS equation which admits exact solutions.

Appendix. The integrability conditions for the potential KdV and KdV hierarchies

This appendix is devoted to the presentation of the integrability conditions for the potential KdV and KdV hierarchies we used in our derivation in section 3. Thus, we shall use the same notation.

The potential KdV hierarchy. The integrable hierarchy of the potential KdV equation is given in equation (23). The quantities $K_j[\phi^{(1)}]$ and their corresponding linearizations $K'_j[\phi^{(1)}]\psi$, for $j = 2$ and $j = 3$, read (here $\partial = \partial_x$):

$$K_2[\phi^{(1)}] = \alpha_1 \partial^3 \phi^{(1)} + \alpha_2 (\partial \phi^{(1)})^2, \tag{A.1}$$

$$K_3[\phi^{(1)}] = \beta_3 \left\{ \partial^5 \phi^{(1)} + \frac{5\alpha_2}{3\alpha_1} \left[\frac{2\alpha_2}{3\alpha_1} (\partial \phi^{(1)})^3 + \left(\partial^2 \phi^{(1)}\right)^2 + 2\partial\phi^{(1)} \partial^3 \phi^{(1)} \right] \right\}, \tag{A.2}$$

and

$$K'_2[\phi^{(1)}]\psi = \alpha_1 \partial^3 \psi + 2\alpha_2 \partial \phi^{(1)} \partial \psi,$$

$$K'_3[\phi^{(1)}]\psi = \beta_3 \left\{ \partial^5 \psi + \frac{10\alpha_2}{3\alpha_1} \left[\partial \phi^{(1)} \partial^3 \psi + \partial^2 \phi^{(1)} \partial^2 \psi + \frac{\alpha_2}{\alpha_1} (\partial^2 \phi^{(1)})^2 \partial \psi + \partial^3 \phi^{(1)} \partial \psi \right] \right\},$$

where α_1, α_2andβ_3 are the real coefficients (in our case they are polynomial functions of the parameter h).

The non-homogeneous terms $f^{(t_2)} \in \mathcal{P}_6^{(1)}$, $f^{(t_3)} \in \mathcal{P}_8^{(1)}$, given in equations (25) and (26), respectively, are

$$f^{(t_2)} = a_1 (\partial \phi^{(1)})^3 + a_2 \partial \phi^{(1)} \partial^3 \phi^{(1)} + a_3 (\partial^2 \phi^{(1)})^2, \tag{A.3}$$

$$\begin{aligned} f^{(t_3)} = {} & b_1 \partial \phi^{(1)} (\partial^2 \phi^{(1)})^2 + b_2 \partial \phi^{(1)} \partial^5 \phi^{(1)} + b_3 \partial^2 \phi^{(1)} \partial^4 \phi^{(1)} \\ & + b_4 (\partial \phi^{(1)})^4 + b_5 (\partial \phi^{(1)})^2 \partial^3 \phi^{(1)} + b_6 (\partial^3 \phi^{(1)})^2. \end{aligned} \tag{A.4}$$

The compatibility condition (27) implies the following algebraic relations between the coefficients a_1, a_2, a_3 and $b_1, \ldots, b_6$:

$$\begin{cases} 9\alpha_1^2 b_1 = 5\beta_3\left[9a_1\alpha_1 + 2\left(a_2 + 3a_3\right)\alpha_2\right], \\ 3\alpha_1 b_2 = 5\beta_3 a_2, \\ 3\alpha_1 b_3 = 5\beta_3\left(a_2 + 2a_3\right), \\ 54\alpha_1^3 b_4 = 5\beta_3\alpha_2\left(27a_1\alpha_1 - a_2\alpha_2\right), \\ 9\alpha_1^2 b_5 = 5\beta_3\left(9a_1\alpha_1 + 5a_2\alpha_2\right), \\ 3\alpha_1 b_6 = 5\beta_3\left(a_2 + a_3\right). \end{cases} \tag{A.5}$$

The system (A.5) allows us to express b_i s as functions of a_i s without requiring any constraints on the latter ones. This means that the compatibility condition (27) is satisfied for any a_1, a_2, a_3 provided that (A.5) is fulfilled.

The non-homogeneous terms $h^{(t_2)} \in \mathcal{P}_8^{(2)}$, $h^{(t_3)} \in \mathcal{P}_{10}^{(2)}$, defined in equations (31) and (32), respectively, are quite long, since $\dim \mathcal{P}_8^{(2)} = 11$ and $\dim \mathcal{P}_{10}^{(2)} = 24$. In order to present the integrability conditions imposed by the compatibility equation (33), it is sufficient to note only the expression of $h^{(t_2)}$. It reads

$$\begin{aligned} h^{(t_2)} = {} & c_1(\partial^3\phi^{(1)})^2 + c_2\partial^2\phi^{(1)}\partial^4\phi^{(1)} + c_3\partial\phi_x^{(1)}\partial^5\phi^{(1)} + c_4\partial\phi^{(1)}\left(\partial^2\phi^{(1)}\right)^2 \\ & + c_5(\partial\phi^{(1)})^2\partial^3\phi^{(1)} + c_6(\partial\phi^{(1)})^4 + c_7\partial\phi^{(1)}\partial^3\phi^{(2)} + c_8\partial^2\phi^{(1)}\partial^2\phi^{(2)} \\ & + c_9\partial^3\phi^{(1)}\partial\phi^{(2)} + c_{10}(\partial\phi^{(1)})^2\partial\phi^{(2)} + c_{11}(\partial\phi^{(2)})^2. \end{aligned} \tag{A.6}$$

The compatibility condition (33) allows one to express the 24 coefficients of $h^{(t_3)}$ in terms of those of $h^{(t_2)}$ (these algebraic relations are easy to derive but rather cumbersome so they are not presented here) and the following three algebraic constraints involving the coefficients $c_1, \ldots, c_{11}$ and the coefficients $a_1, a_2, a_3, \alpha_1, \alpha_2$ previously defined. The obtained necessary integrability conditions read

$$\begin{aligned} c_6 = {} & \frac{\left[27a_1\left(a_2 + 4a_3\right)\alpha_1 - \left(37a_2^2 + 46a_2a_3 + 12a_3^2\right)\alpha_2\right]c_{11}}{108\alpha_1^2\alpha_2} \\ & + \frac{\left[\left(17a_2 + 18a_3\right)\alpha_2 - 27a_1\alpha_1\right]c_8}{108\alpha_1^2} + \frac{\left[\left(3a_2 - 8a_3\right)\alpha_2 - 3a_1\alpha_1\right]c_9}{36\alpha_1^2} \\ & + \frac{\left(18c_2 - 24c_1 - 55c_3\right)\alpha_2^2}{54\alpha_1^2} + \frac{\left(13c_5 - 3c_4\right)\alpha_2}{18\alpha_1}, \\ c_7 = {} & \frac{a_2c_{11}}{\alpha_2}, \\ c_{10} = {} & \frac{3a_1c_{11}}{\alpha_2}. \end{aligned}$$

The KdV hierarchy. For the integrable hierarchy of the KdV equation, we have

$$H_2[\varphi^{(1)}] = \alpha_1\partial^3\varphi^{(1)} + 2\alpha_2\varphi^{(1)}\partial\varphi^{(1)},$$

$$H_3[\varphi^{(1)}] = \beta_3\left\{\partial^5\varphi^{(1)} + \frac{10\alpha_2}{3\alpha_1}\left[\frac{\alpha_2}{\alpha_1}\left(\varphi^{(1)}\right)^2\partial\varphi^{(1)} + 2\partial\varphi^{(1)}\partial^2\varphi^{(1)} + \varphi^{(1)}\partial^3\right]\right\},$$

and

$$H_2'[\varphi^{(1)}]\rho = \alpha_1\partial^3\varphi^{(2)} + 2\alpha_2(\rho\partial\varphi^{(1)} + \varphi^{(1)}\partial\rho),$$

$$H_3'[\varphi^{(1)}]\rho = \beta_3\left\{\partial^5\rho + \frac{10\alpha_2}{3\alpha_1}\left[\varphi^{(1)}\partial^3\rho + 2\partial\varphi^{(1)}\partial^2\rho + \left(2\partial^2\varphi^{(1)} + \frac{\alpha_2}{\alpha_1}(\varphi^{(1)})^2\right)\partial\rho \right.\right.$$
$$\left.\left. + \left(\frac{2\alpha_2}{\alpha_1}\varphi^{(1)}\partial\varphi^{(1)} + \partial^3\varphi^{(1)}\right)\rho\right]\right\}.$$

The above expressions are obtained by differentiating wrt x the corresponding expressions given in equations (A.1)–(A.2) and setting $\varphi^{(1)} = \partial_x\phi^{(1)}$, $\rho = \partial_x\psi$.

The non-homogeneous terms $g^{(t_2)} \in \mathcal{P}_9^{(2)}$, $g^{(t_3)} \in \mathcal{P}_{11}^{(2)}$, defined in equations (28) and (29), respectively, are quite long, since $\dim\mathcal{P}_9^{(2)} = 14$ and $\dim\mathcal{P}_{11}^{(2)} = 31$. In order to present the integrability conditions imposed by the compatibility equation (30), it is sufficient to write down only the expression $g^{(t_2)}$. It reads

$$\begin{aligned} g^{(t_2)} = {} & d_1\partial^2\varphi^{(1)}\partial^3\varphi^{(1)} + d_2\partial\varphi^{(1)}\partial^4\varphi^{(1)} + d_3\varphi^{(1)}\partial^5\varphi^{(1)} + d_4(\partial\varphi^{(1)})^3 \\ & + d_5\varphi^{(1)}\partial\varphi^{(1)}\partial^2\varphi^{(1)} + d_6(\varphi^{(1)})^2\partial^3\varphi^{(1)} + d_7(\varphi^{(1)})^3\partial\varphi^{(1)} + d_8\varphi^{(1)}\partial^3\varphi^{(2)} \\ & + d_9\partial\varphi^{(1)}\partial^2\varphi^{(2)} + d_{10}\partial^2\varphi^{(1)}\partial\varphi^{(2)} + d_{11}\varphi^{(2)}\partial^3\varphi^{(1)} + d_{12}(\varphi^{(1)})^2\partial\varphi^{(2)} \\ & + d_{13}\varphi^{(1)}\varphi^{(2)}\partial\varphi^{(1)} + d_{14}\varphi^{(2)}\partial\varphi^{(2)}. \end{aligned} \tag{A.7}$$

The compatibility condition (30) allows us to express 31 coefficients of $g^{(t_3)}$ in terms of those of $g^{(t_2)}$ (these algebraic relations are easy to derive but rather cumbersome and we do not present them here) and the following 5 integrability conditions involving the coefficients $d_1, \ldots, d_{14}$ and the coefficients $a_1, a_2, a_3, \alpha_1, \alpha_2$ previously defined:

$$\begin{aligned} d_7 = {} & \frac{\left[9a_1(12a_3 + 5a_2)\alpha_1 - \left(45a_2^2 + 88a_2a_3 + 12a_3^2\right)\alpha_2\right]d_{14}}{54\alpha_1^2\alpha_2} \\ & + \frac{[(3a_2 - 8a_3)\alpha_2 - 3a_1\alpha_1]d_{10}}{9\alpha_1^2} + \frac{2[(21a_3 + 4a_2)\alpha_2 - 9a_1\alpha_1]d_9}{27\alpha_1^2} \\ & + \frac{(9d_5 + 8d_6 - 24d_4)\alpha_2}{9\alpha_1} - \frac{2(12d_1 - 30d_2 + 85d_3)\alpha_2^2}{27\alpha_1^2}, \\ d_8 = {} & \frac{a_2d_{14}}{2\alpha_2}, \\ d_{11} = {} & d_{10} - d_9 + \frac{a_2d_{14}}{2\alpha_2}, \\ d_{12} = {} & \frac{3a_1d_{14}}{2\alpha_2}, \\ d_{13} = {} & \frac{3a_1d_{14}}{\alpha_2}. \end{aligned}$$

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